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Generalized Isothermal Universes

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We present a simple nonstatic generalization of the isothermal universe. The cosmological fluid obeys a barotropic equation of state of the form $p = \alpha\rho$. We employ a causal heat transport equation of the Maxwell–Cattaneo form to study the thermodynamical behavior of our model.

KEY WORDS: gravitational clustering; early universe; thermodynamics.

1. INTRODUCTION

Inhomogeneous cosmological models with either heat flux or shear viscosity seem to have solved many of the pathologies of the standard model such as the horizon problem, entropy generation, and the flatness problem (Deng and Mannheim, 1990, 1991) without invoking exotic matter distributions such as scalar fields or dark matter. However, many of the models incorporating heat flux lack a proper thermodynamical investigation especially when the cosmic fluid is far from equilibrium (Anile *et al.*, 1998; Krasinski, 1997). Causal heat transport was first investigated by Triginer and Pavon (1995) in spherically symmetric inhomogeneous cosmologies. This was later followed by other workers (Govender and Govinder, 2001; Herrera *et al.*, 2000; Maartens *et al.*, 1999) with many interesting results such as inflationary models driven by causal heat flux.

In this paper we present a simple method which generalizes the static isothermal universe first studied by Saslaw *et al.* (1996). The Saslaw *et al.* model has many interesting features. Firstly, the cosmic fluid in the static model obeys a barotropic equation of state of the form $p = \alpha\rho$. This well defined equation of state together with simple forms for the metric functions allows an exact treatment of the dynamical quantities associated with the matter distribution. Secondly, it has been argued that the isothermal cosmological model of Saslaw *et al.* could represent the asymptotic state of the Einstein–de Sitter model. The generalized

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model incorporates many of the essential features of the static isothermal universe such as the barotropic equation of state. In fact at late times the generalized model evolves to the static model of Saslaw *et al*. The generalized model could describe an isothermal sphere of galaxies in quasi-hydrostatic equilibrium with heat dissipation driving the system to equilibrium. As pointed in Saslaw *et al.* (1996), any local inhomogeneities that develop within such a system of galaxies would damp out as a result of the nonlinear interactions with surrounding galactic matter. Such a distribution would still be an isothermal sphere with the characteristic r^{-2} density distribution.

The paper is organized as follows. In Section 1, we present the static isothermal universe first investigated by Saslaw *et al.* (1996). In Section 2, we generalize the static model to include heat flux. In the next section a thermodynamical treatment within the context of extended irreversible thermodynamics is carried out. The special case of inflation is considered in Section 4.

2. THE BASIC EQUATIONS

We begin with a general spherically symmetric static metric of the form

$$ds^2 = -A_0^2(r)\,dt^2 + B_0^2(r)\,dr^2 + r^2\,d\Omega^2, \tag{1}$$

where $d\Omega^2$ is the metric of the unit 2-sphere. The static isothermal universe due to Saslaw *et al.* (1996) is characterised by

$$A_0 = Ar^{4\alpha/(1+\alpha)}, \qquad B_0 = 1 + \frac{4\alpha}{(1+\alpha)^2}, \tag{2}$$

where A is an arbitrary constant. The energy density and pressure for the static model is given by

$$\rho_0 = \frac{4\alpha}{4\alpha + (1+\alpha)^2}\frac{1}{r^2} \tag{3}$$

$$p_0 = \frac{4\alpha^2}{4\alpha + (1+\alpha)^2}\frac{1}{r^2}, \tag{4}$$

which clearly indicates that the fluid obeys a barotropic equation of state of the form $p = \alpha\rho$. This model was shown to be dynamically stable under density perturbations owing to the fact that it is always in hydrostatic equilibrium. Any perturbation in the density profile will be accompanied by a corresponding pressure change that keeps the cosmic fluid in hydrostatic equilibrium. We observe that, while both ρ_0 and P_0 diverge as $r \to 0$, the mass contained within a shell of radius R

$$m = \int_0^R \rho_0 r^2\,dr \tag{5}$$

remains finite.

In attempting to find a nonstatic generalization of the static isothermal universe given by (1) we begin with a spherically symmetric shear-free metric of the form

$$ds^2 = -A_0^2(r)\,dt^2 + b^2(t)\left[B_0^2(r)\,dr^2 + r^2\,d\Omega^2\right], \tag{6}$$

where $A_0(r)$ and $B_0(r)$ are given by (2) and the metric function $b(t)$ is yet to be determined. This form of the metric has been used extensively in the past to effectively model the gravitational collapse of radiating stars with an initial static matter distribution (Bonner *et al.*, 1989).

The fluid 4-velocity is $u^a = A_0^{-1}\delta_0^a$, its volume rate of expansion $\Theta = \nabla_a u^a$ is

$$\Theta = \frac{1}{A_0}\frac{\dot{b}}{b} \tag{7}$$

where an overdot denotes d/dt.

We model the matter content as an imperfect fluid with heat conduction so that the energy–momentum tensor is (Bonner *et al.*, 1989)

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a, \tag{8}$$

where ρ is the energy density, p is isotropic pressure, and $q_a = q n_a$ is the heat flux, with n^a a unit radial vector.

The Einstein field equations for the metric (6) and energy momentum tensor (8) yield

$$\rho = \frac{1}{b^2}\left(\frac{3}{A_0^2}\dot{b}^2 + \rho_0\right) \tag{9}$$

$$p = \frac{1}{b^2}\left[-\frac{1}{A_0^2}(2\ddot{b}b + \dot{b}^2) + p_0\right] \tag{10}$$

$$q = -\frac{2}{A_0 B_0 b}\left(\frac{A_0'}{A_0}\frac{\dot{b}}{b}\right), \tag{11}$$

where we have introduced

$$\rho_0 = -\frac{1}{B_0^2}\left[\frac{1}{r^2} - \frac{2}{r}\frac{B_0'}{B_0}\right] + \frac{1}{r^2}, \tag{12}$$

$$p_0 = \frac{1}{B_0^2}\left[\frac{1}{r^2} + \frac{2}{r}\frac{A_0'}{A_0}\right] - \frac{1}{r^2}, \tag{13}$$

which define the energy density and pressure of the static matter configuration. Since A_0 and B_0 represent the static solution, the pressure isotropy condition

$$\frac{A_0''}{A_0} - \frac{A_0'}{A_0}\frac{B_0'}{B_0} - \frac{1}{r}\left(\frac{1}{r} + \frac{A_0'}{A_0} + \frac{B_0'}{B_0}\right) + \frac{B_0^2}{r^2} = 0 \tag{14}$$

is immediately satisfied. To close the system of equations we impose a barotropic equation of state of the form $p = \alpha\rho$ (where α is a constant) which leads to

$$2b\ddot{b} + (1 + 3\alpha)\dot{b}^2 = A_0^2(p_0 - \alpha\rho_0). \tag{15}$$

Since the static isothermal universe also obeys a barotropic equation of state ($p_0 = \alpha\rho_0$), we have from (15)

$$2b\ddot{b} + (1 + 3\alpha)\dot{b}^2 = 0 \tag{16}$$

of which the general solution is

$$b(t) = (c_0 t + c_1)^{2/3(1+\alpha)} \tag{17}$$

with c_0 and c_1 constants of integration. We also note that (16) admits inflationary solutions when

$$b(t) = b_0\, e^{H_0 t}, \tag{18}$$

where b_0 and $H_0 > 0$ are constants. In this case we must have $\alpha = -1$.

3. INHOMOGENEOUS COSMOLOGICAL MODEL

As an application in cosmology we will consider our solution (17) more closely. The metric (6) can be written as

$$ds^2 = -A^2 r^{8\alpha/(1+\alpha)}\, dt^2 + (c_0 t + c_1)^{4/3(1+\alpha)}\left[\left(1 + \frac{4\alpha}{(1+\alpha)^2}\right)^2 dr^2 + r^2\, d\Omega^2\right]. \tag{19}$$

The dynamical quantities for the nonstatic generalization are

$$\rho = \left[\frac{4\alpha}{4\alpha + (1+\alpha)^2}\frac{1}{r^2}\right](c_0 t + c_1)^{-4/3(1+\alpha)} + \frac{3c_0^2}{3A^2(1+\alpha)^2}(c_0 t + c_1)^{-2} \tag{20}$$

$$p = \left[\frac{4a^2}{4\alpha + (1+\alpha)^2}\frac{1}{r^2}\right](c_0 t + c_1)^{-4/3(1+\alpha)} + \frac{3\alpha c_0^2}{3A^2(1+\alpha)^2}(c_0 t + c_1)^{-2} \tag{21}$$

$$q = -\frac{8a(1+a)r^{-(1+5\alpha)/(1+\alpha)}}{A(\alpha^2 + 6\alpha + 1)}(c_0 t + c_1)^{-(5+3\alpha)/3(1+\alpha)}. \tag{22}$$

In order to investigate the physical viability of the models we employ a causal heat transport equation of Maxwell–Cattaneo type

$$\tau h_a^b u^c \nabla_c q_b + q_a = -\kappa\left(h_a^b \nabla_b T + T u^b \nabla_b u_\alpha\right), \tag{23}$$

where $h_a^b = \delta_a^b + u_a u^b$ is the projection tensor into the comoving rest space, T is the local equilibrium temperature, κ is the thermal conductivity, and τ is the relaxational timescale. Setting $\tau = 0$ in (23) we obtain the noncausal Fourier–Eckart law for quasi-stationary heat transport, which leads to pathological behavior in the propagation velocity of thermal signals.

For the general metric (19), Eq. (23) reduces to

$$\tau_r \dot{q} + A_0 q = -\frac{\kappa}{B_0 b}(A_0 T)', \tag{24}$$

where τ_r is the relaxation time. We assume that the matter content behaves like a radiation fluid interacting with matter and hence the thermal conductivity κ is given by

$$\kappa = \kappa_0 T^3 \tau_c, \tag{25}$$

where τ_c is the mean collision time. We note that the expansion of the universe defines a natural timescale, the expansion time, $H = 3|\Theta|^{-1}$. A necessary condition for maintaining local thermal equilibrium between electrons, baryonic matter, and radiation is

$$\tau_c < H^{-1}. \tag{26}$$

For example, at a high enough temperature of the order of 10^{10} K, as is the case for neutrino decoupling we have

$$\tau_c \propto H^{-5}, \quad H \propto T^2. \tag{27}$$

At lower temperatures of the order of 10^3 K, where electron–photon interactions occur via Thompson scattering we may assume that the relaxation time is proportional to the collision time, τ_c:

$$\tau_r = \beta \tau_c, \tag{28}$$

where β is a constant. This is only a first approximation as a physically viable assumption would be

$$\tau_r = \frac{3\gamma(t)}{\Theta}, \tag{29}$$

where $\gamma(t)$ acts as a fine-tuning parameter (Coley *et al.*, 2002).

We further assume that the cosmic fluid is composed of highly relativistic particles, as was the case in the very early universe when dissipative processes took place. Here we take the energy density to be

$$\rho = aT^4, \tag{30}$$

where a is a constant. Substituting all of this into (24) yields

$$\tau_c = \frac{2A_0'}{A_0 B_0}\frac{\dot{b}}{b^2}\left[2\frac{\beta A_0'}{A_0^2 B_0}\left(\frac{\ddot{b}}{b} - 2\frac{\dot{b}^2}{b^2}\right) - \frac{\kappa_0 A_0 \rho_0}{a B_0 b^2}\left(\frac{A_0'}{A_0} + \frac{\rho_0'}{4\rho_0}\right) - 3\frac{\kappa_0}{2aB_0 A_0^2}\left(\frac{\dot{b}}{b}\right)^2\right]^{-1}. \tag{31}$$

In order for the particles to remain in thermal equilibrium with the cosmic fluid we must have

$$\tau_c < |\Theta|^{-1}. \tag{32}$$

Since we assume here that the cosmic fluid behaves irreversibly, the entropy is no longer conserved, but increases, according to the second law of thermodynamics (Maartens, 1996). The entropy production rate is given by the divergence of the entropy four-current and since we have a nonzero heat flux we can write

$$S^a_a = \frac{q^q q_a}{\kappa T^2}, \tag{33}$$

where the thermodynamical temperature is given in (30). The deviation from equilibrium is measured by the covariant dimensionless ratio

$$\frac{|q|}{\rho} = 2\frac{A_0'}{B_0\left(3\dot{b}^2 + \rho_0 A_0^2\right)}|\dot{b}|. \tag{34}$$

For the radiation dominated era ($\alpha = 1/3$), this ratio decays rapidly at late times (see Fig. 5). More interestingly, this ratio is independent of r for all time during the radiation dominated era. This implies that the fluid is only close to equilibrium at late times as one expects.

Figures 1 and 2 compare the collision time derived from the Eckart theory and the extended irreversible thermodynamics scenario respectively to the Hubble time for the radiation dominated era. It is clear that the Eckart theory displays pathological behavior of the collision time, especially during the early evolution of

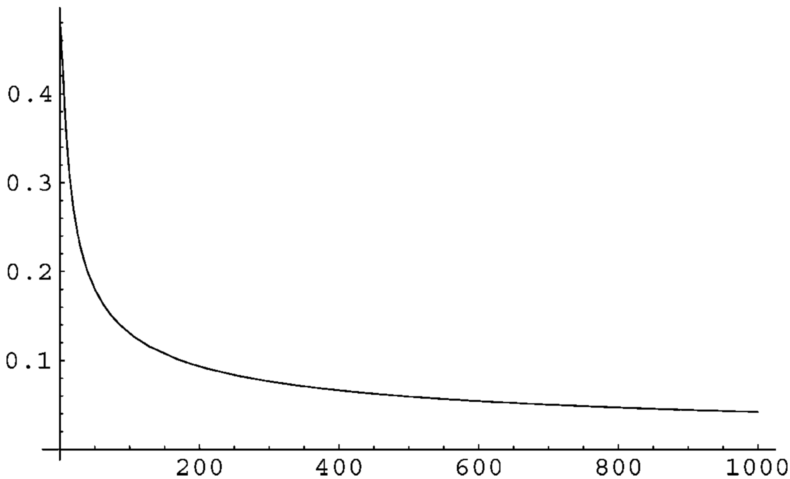

Fig. 5. $|q|^p$ vs. time, with $\alpha = 1/3$.

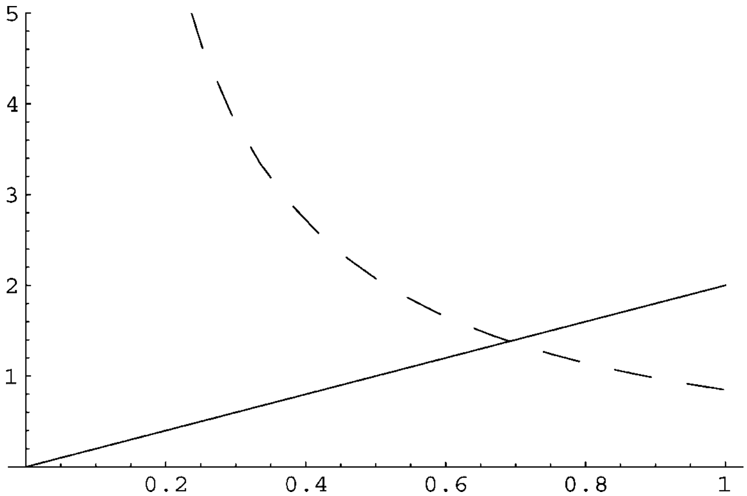

Fig. 1. Comparison of Hubble time (solid line) and "noncausal" collision time ($\beta = 0$) (broken line) with $\alpha = 1/3$ and $r = 1$.

the model. One expects this as the cosmic fluid in this period is far from equilibrium, and relaxational effects dominate here (Anile *et al.*, 1998).

Figures 3 and 4 depict the entropy production rate in the Eckart and extended irreversible thermodynamics cases, respectively. We note that the entropy production rate is of many orders of magnitude greater in the causal theory as compared to the Eckart theory for early evolution times. As the model evolves

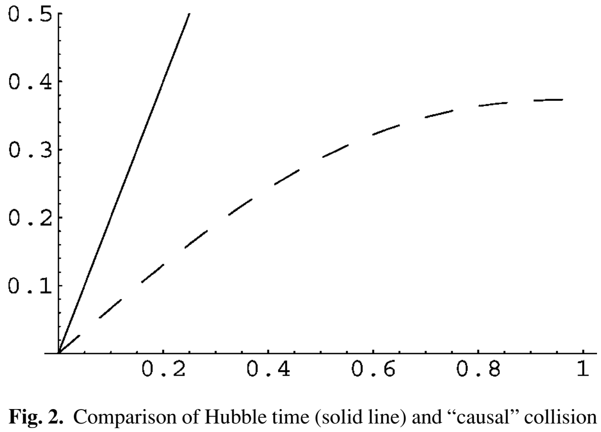

Fig. 2. Comparison of Hubble time (solid line) and "causal" collision time ($\beta = 1$) (broken line) with $\alpha = 1/3$ and $r = 1$.

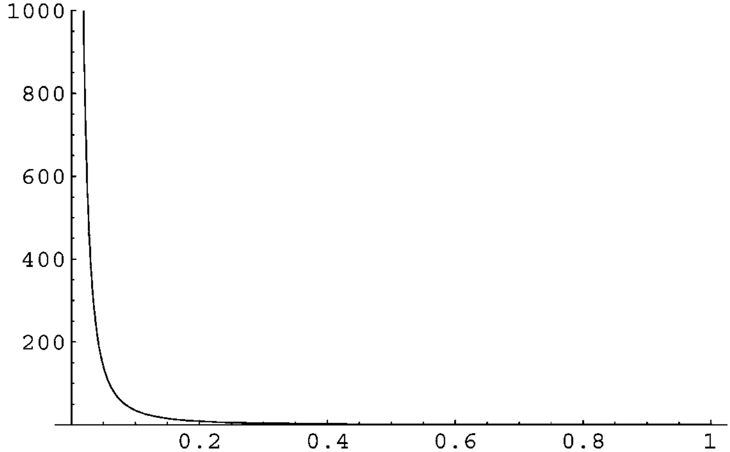

Fig. 3. Noncausal entropy production, (33), using (2) with $\alpha = 1/3$ and $r = 1$.

the entropy production rate drops off and for late times the Eckart and extended thermodynamical models predict similar behavior.

We also note that when $\alpha = 0$, our model reduces to the flat FRW model, i.e., the Einstein–de Sitter universe. Here both the pressure and the heat flux vanishes and the energy density behaves as t^{-2}. We then conclude that the matter dominated, nonstatic isothermal universe, is the Einstein–de Sitter universe.

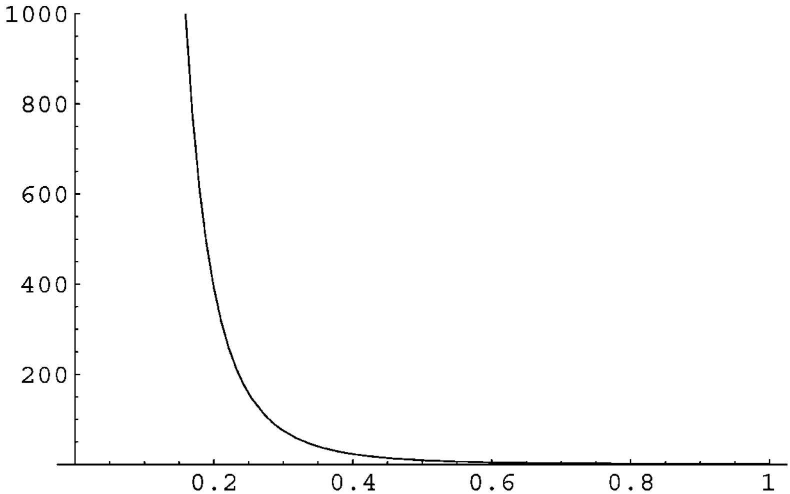

Fig. 4. Causal entropy production, (33), using (2) with $\alpha = 1/3$ and $r = 1$.

4. INFLATIONARY COSMOLOGY

In this section we employ the causal heat transport equation (23) to investigate the effects of heat dissipation in an inflating model. We use the solution (18) found in Section 2. The Einstein field equations for the solution (18) yield

$$\rho = \frac{\rho_0}{b_0^2} e^{-2H_0 t} + \frac{3H_0^2}{A_0^2} \tag{35}$$

$$p = \frac{p_0}{b_0^2} e^{-2H_0 t} - \frac{3H_0^2}{A_0^2} \tag{36}$$

$$q = -\frac{2A_0'}{A_0^2 B_0} \left(\frac{H_0}{a_0} \right) e^{-H_0 t} \tag{37}$$

It is clear that for late times the model approaches a de Sitter-like equilibrium with $p = -\rho$. The causal heat transport equation (23) reduces to

$$2\frac{A_0'}{A_0^2} \frac{H_0}{a_0} (A_0 - H_0 \tau) = \frac{\kappa (A_0 T)'}{b_0}, \tag{38}$$

where the collision time, thermal conductivity, and temperature profile are yet to be specified. Following Maartens *et al.* (1999) we set

$$\tau = A_0 H_0^{-1} \tag{39}$$

$$T = \frac{U(t)}{A_0}, \tag{40}$$

where $U(t)$ is an arbitrary positive function and the relaxational time is of the order of the Hubble time. Note that for any choice of U, the temperature decreases radially outward with the heat flux being directly radially inward. This is acceptable as pointed in Maartens *et al.* (1999) and Herrera *et al.* (2000), if one notes that the accelerative contribution to the heat flux (arising from the inertia of the heat energy) dominates over the temperature gradient. It was recently shown that the strength of dissipative inflation can be measured by a control parameter of the form

$$\bar{\alpha} = \frac{\kappa T}{\tau(\rho + p)} \tag{41}$$

in a fluid undergoing heat dissipation only. For our model we have

$$\bar{\alpha} = \frac{\kappa U \, e^{H_0 t} b_0^2 \Theta}{\rho_0 (1 + \alpha) A_0}, \tag{42}$$

where the fluid volume expansion is given by

$$\Theta = 3\frac{H_0}{A_0}. \tag{43}$$

We note that stronger inflationary expansion is characterized by larger values of Θ, which confirms earlier findings that the parameter $\bar{\alpha}$ can be utilized to measure the strength of dissipative inflation (Govender and Govinder, 2001; Herrera *et al.*, 2000).

5. SUMMARY

We have provided a simple framework which generalizes the static cosmological model of Saslow *et al.* to include the effects of heat dissipation. The models presented here are simple exact models, which are at least not physically unreasonable in describing some epoch of the evolution of the early universe. Such models allow for a more transparent analysis of the underlying physics, and they can also serve as a useful check for numerical procedures that arise in more realistic solutions.

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